Non-vanishing

Recall :

Theorem B (Special finiteness)

- X proj normal, $\mathbb{Q}$-factorial
- $V \subseteq \mathrm{WDiv}_{\mathbb{R}}(X)$ finite dimensional, defined $/\mathbb{Q}$
- (X, Δ_0) dlt $\quad S \leq \Delta_0$
- $\mathcal{D}$ fixed set of divisors.

Then there exist finitely many birational maps $\phi_i : X \dashrightarrow Y_i$ st if $\phi : X \dashrightarrow Y$ is weak log canonical model of $K_X + \Delta$, $\Delta \in \mathcal{L}_{S+A}(V)$ which only contracts elements of $\mathcal{D}$ and does not contract S, there is $\xi_i : Y_i \dashrightarrow Y$ iso near S.

Theorem C (Existence of log terminal models)

Suppose

$K_X+\Delta$ klt, Δ big, $K_X+\Delta \sim_{\mathbb{R}} D \geq 0$

then $K_X+\Delta$ has a log terminal model.

Theorem D (Non vanishing)

Suppose

$K_X+\Delta$ klt, Δ big,

$K_X+\Delta$ is pseudoeffective, then

$K_X+\Delta \sim_{\mathbb{R}} D \geq 0.$

Today's goal

$$D_{n-1} + B_n + C_n \Rightarrow D_n$$

Sketch of proof

Idea: Try to mimic the proof of the standar non-vanishing theorem (Kollár - Mori)

Standard non-vanishing

We have a big and net divisor.
We use bigness to create a highly singular section $\Rightarrow$ a lc center, we use induction + Kawamata Viehweg vanishing to lift that section.

Main problems.

1) Δ is only big

2) Kawamata - Viehweg vanishing only applies to $\mathbb{Q}$-divisors, whereas we are dealing with $\mathbb{R}$ divisors.

Both of these issues are solved by the following idea: throw at Δ an ample divisor H.

$$\underbrace{K_X + \Delta + H}_{\text{big}} \xrightarrow[\text{by special finiteness}]{} \text{finitely many minimal models and recover netness.}$$

We want to take the limit as $t \to 0$ of $K_X + \Delta + tH$

We do this by relating the minimal models of small perturbations of Δ with the minimal models of $K_X+\Delta+H$.

$$K_X+\Delta \sim \sum r_i (K_X+\Delta_i) \sim \sum r_i D_i$$

$\mathbb{Q}$-divisors

Lemma Assume C_n

(X, Δ) $\qquad \lfloor \Delta \rfloor = 0$

$K_X + \Delta$ is pseudoeffective

$\Delta - A \geq 0 \qquad A$ ample $\mathbb{Q}$-div

Suppose for each k,

$$h^0(X, \mathcal{O}_X(\lfloor mk(K_X+\Delta) \rfloor + kA)) \overset{(*)}{=} O(1)$$

in m.

Then $\exists D$ st $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

Pf $(*)$ implies $K_X + \Delta \equiv N_\sigma(K_X+\Delta) \geq 0$

If $K_X + \Delta \sim_{\mathbb{R}} N_\sigma(K_X+\Delta)$, we would be done.

We try to get close to

$\mathbb{R}$ - linear equivalence

$$A'' \stackrel{\text{def}}{=} \overset{\text{ample}}{A} + \underbrace{N_\sigma(K_X+\Delta) - (K_X+\Delta)}_{\text{numerically trivial}}$$

is ample

$$\exists\, A' \sim_{\mathbb{R}} A''$$

$$\Delta' = A' + \underbrace{(\Delta - A)}_{\geq 0 \text{ by hypothesis}} \sim_{\mathbb{R}} \Delta$$

- $K_X + \Delta'$ is klt
- $K_X + \Delta' \equiv K_X + \Delta$
- $A' - A \sim_{\mathbb{R}} N_\sigma(K_X+\Delta) - (K_X+\Delta)$

$$K_X + \Delta' = K_X + A' + (\Delta - A)$$

$$\sim_{\mathbb{R}} \cancel{K_X} + N_\sigma(K_X+\Delta) - \cancel{(K_X+\Delta)} + \cancel{\Delta}$$

$$\underbrace{K_X+\Delta'}\sim_{\mathbb{R}} N_\sigma(K_X+\Delta) = N_\sigma(\underbrace{K_X+\Delta'})$$

By C_n $K_X+\Delta'$ has a log terminal model, and therefore $K_X+\Delta'$ is effective by the basepoint free theorem.

C_n : $K_X+\Delta'$ klt ✓

Δ' big ✓

$K_X+\Delta' \sim_{\mathbb{R}} D' \geq 0$ ✓

$\Rightarrow$ exist log terminal model of $K_X+\Delta'$.

This is also a log terminal model of $K_X+\Delta$. Now use basepoint free theorem.

So from now on we assume we have plenty of sections.

<u>Lemma</u>

$(X, \Delta = A + B)$ $\quad A$ ample $\quad \lfloor B \rfloor = 0$

Suppose

$h^0(X, \mathcal{O}_X(\lfloor mk(K_X+\Delta)\rfloor + kA)) \neq O(1)$ in m

then, there is (Y, Γ) st

- Y is birational to X
- $\Gamma - C \geq 0$ $\quad C$ ample
- $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ $\not\exists D$ $K_Y + \Gamma \sim_{\mathbb{R}} G > 0$

(So far we may take (X, Δ))

- $T = \lfloor \Gamma \rfloor$ is an irreducible divisor
- Γ and $N_\sigma(K_Y + \Gamma)$ have no common components.

Pf

First we'd like to create an lc center. Let m be so large that

$$h^0(X, \mathcal{O}_X(\lfloor mk(K_X+\Delta)\rfloor + kA)) > \binom{kn+n}{n}$$

$\exists\, H \sim_{\mathbb{R}} m(K_X+\Delta) + A$

$\mathrm{mult}_x H > n.$

$x \in \mathrm{NKLT}(X, H).$

Slight problem: H is of mixed type.

Solution: Scale it.

$$(t+1)(K_X+\Delta) \sim_{\mathbb{R}} K_X + \underbrace{\frac{m-t}{m} A + B + \frac{t}{m} H}_{\Delta_t}$$

So far:

1) $K_X + \Delta_0 = K_X + \Delta$ is KH

2) $\Delta_t \geq A' = \frac{\varepsilon}{m} A$ if $t \leq m - \varepsilon$

3) $x \in \mathrm{NKlt}(X, \Delta_{m-\varepsilon} = K_X + \varepsilon A' + \delta + (1-\varepsilon) H)$

Let $\pi : Y \to X$ be a log resolution of $(X, \Delta + H)$ then we may write

$$K_Y + \Psi_t = \pi^*(K_X + \Delta_t) + E_t$$

Key thing: $\lfloor \Psi_t \rfloor \neq \emptyset$

Idea to get (Y, Γ) we work on (Y, Ψ_t)

Pick F exceptional $F \geq 0$ so that $\pi^* A' - F$ is ample.

Pick $C \in |\pi^* A' - F|_{\mathbb{Q}}$

$$\Phi_t = \Psi_t - \pi^* A' + C + F \geq 0.$$

$$\Gamma_t = \Phi_t - \Phi_t \wedge N_\sigma(K_Y + \Phi_t)$$

Lemma X normal, D_1, D_2, D_3 $\mathbb{R}$-Cartier.

$f_i : X \dashrightarrow Z_i$ ample models for D_i

Suppose f_i are birational does not contract a certain S.

Assume $Z_1 \dashrightarrow Z_2$ iso near S.

$$\begin{tikzcd}
 & X \arrow[dl, dashed] \arrow[d, dashed] \arrow[dr, dashed] & \\
Z_1 \arrow[r, dashed, "\text{iso near } S"'] \arrow[rr, bend left, dashed] & Z_2 & Z_3
\end{tikzcd}$$

If $D_3 = \lambda_1 D_1 + \lambda_2 D_2$ $\lambda_i \geq 0$.

then $Z_1 \dashrightarrow Z_3$ is iso near S

Pf It's easy to replace f_1, f_3 by morphisms. Then one checks that near S f_1 contracts exactly the same curves as f_3.

Lemma Assume $B_n + C_n$.

Suppose: $D_{n-1} + B_n + C$

- $(X, \Delta_0 = S + A + B_0)$ log smooth.
- $K_X + \Delta_0$ is pseudo effective.
- S is not a component of $N_\sigma(K_X + \Delta_0)$
- V_0 finite dimensional affine space containing B_0, defined $/\mathbb{Q}$

Then there exists an ample divisor H, a log terminal model $\phi : X \dashrightarrow Y$ of $K_X + \Delta_0 + H$ and a constant α st if $B \in V_0$, $\|B - B_0\| < \alpha t$ then there is a log terminal model

$\varphi : X \dashrightarrow Z$ of $K_X + \underbrace{S + A + tH}_{\Delta}$

that does not contract

S and, $Y \dashrightarrow Z$ is iso

near S.

<u>Pf</u> Pick general ample $\mathbb{Q}$-divisors

$H_1, \dots, H_k$ that span $NS_{\mathbb{R}}(X)$

V = Affine span of V_0, $H_1, \dots, H_k$.

Let $\mathcal{C} \subseteq \mathcal{L}_{S+A}(V)$ be a convex

subset st

- $\Delta_0 \notin \mathcal{C}$ (just rescale $\mathcal{C}$)
- $\Delta_0 \in \overline{\mathcal{C}}$

- If $\Delta \in \mathcal{C}$ then $K_X+\Delta$ is plt

$\Delta - \Delta_0$ ample

$$\operatorname{Supp}(\Delta-\Delta_0) = \bigcup_{i=1}^{k} H_i$$

In particular: if $\Delta \in \mathcal{C}$

then $N_\sigma(K_X+\Delta) \le N_\sigma(K_X+\Delta_0)$

(because $K_X+\Delta = \underline{\underline{K_X+\Delta_0}} + \underbrace{(\Delta-\Delta_0)}_{\text{ample}}$

By shrinking $\mathcal{C}$ we may assume

$$\operatorname{Supp} N_\sigma(K_X+\Delta) = \operatorname{Supp} N_\sigma(K_X+\Delta_0)$$

$\Delta \in \mathcal{C} \Rightarrow K_X + \Delta$ is big

$\Rightarrow$ has a lt model by Cu.

$\phi : X \dashrightarrow Y.$

$$Ex(\phi) = \operatorname{Supp} N_\sigma (K_X + \Delta)$$
$$= \operatorname{Supp} N_\sigma (K_X + \Delta_0) \not\ni S$$

$\Rightarrow \phi$ does not contract S.

Given $\phi : X \dashrightarrow Y$ define a subset

$\mathcal{S}_\phi \subseteq \mathcal{C}$ as follows:

$\Delta \in \mathcal{S}_\phi \Leftrightarrow \exists$ log terminal model

$\psi : X \dashrightarrow Z$ of $K_X + \Delta$ st

$\chi : Y \dashrightarrow Z$ is iso near S.

$$\mathcal{C} = \bigcup_{j=1}^{\ell} S_{\phi_j}$$

$\downarrow$

Special finiteness $\quad \mathcal{D} = \operatorname{Supp} N_\sigma(K_X+\Delta)$

$$\mathcal{C} = \overline{\bigcup S^\circ_{\phi_j}}$$

After possibly shrinking we may take $\mathcal{C} = \overline{S^\circ_{\phi_j}}$

Fact S_{ϕ_j} are convex by the previous lemma.